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LETTER TO THE EDITOR

A remark on the classical realization of the double quantum group $SU_q(\eta, J)$

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Abstract. By extending the method of Zhe Chang *et al* to a double complex form, the double quantum group $SU_q(\eta, J)$ is realized in the sense of a Poisson bracket, and the meaning is discussed.

We have studied the hyperbolic complex linear symmetry groups in [1]. The ordinary complex linear Lie groups and the hyperbolic complex linear Lie groups are combined into the double complex linear Lie groups, and they are used in the double complex method [2]. Therefore, a problem is raised: how are the double quantum groups corresponding to the double complex linear Lie groups realized?

Recently, in [3] it was pointed out that the usual quantum groups can be realized at the classical level in the sense of a Poisson bracket; however the results only relate to the algebras concerning the ordinary Lie groups. We find that the method, in fact, can be doubled, i.e. by the double complex function method [2] the usual double quantum groups corresponding to the double complex Lie groups can be realized. In the following we concretely discuss the problem of how to realize the double quantum group $SU_q(\eta, J)$ and discuss the meaning, where the signature is equal to diag$(1, \pm 1)$.

Regarding the general discussion of the hyperbolic complex numbers, see [4]. Here we write the hyperbolic pure imaginary unit as ε, $\varepsilon^2 = +1$ and $\varepsilon \neq \pm 1$. Let J denote the double pure imaginary unit, i.e. $J = \mathrm{i}$ ($\mathrm{i}^2 = -1$), or $J = \varepsilon$. If $\{a_n\} = \{a_0, a_1, \ldots, a_n, \ldots\}$ is a sequence of real numbers obeying $\Sigma_{n=0}^{\infty} |a_n| < +\infty$, then we use $a(J)$ to correspond to the real number pair (a_C, a_H), where $a_C = \Sigma_{n=0}^{\infty} (-1)^n a_n$, $a_H = \Sigma_{n=0}^{\infty} (+1)^n a_n = \Sigma_{n=0}^{\infty} a_n$. Therefore we can directly write $a(J) = \Sigma_{n=0}^{\infty} a_n J^{2n}$, $a_C = a(J = \mathrm{i})$, $a_H = a(J = \varepsilon)$, and we call $a(J)$ a double real number generated by the sequence $\{a_n\}$. If $a(J)$ and $b(J)$ are double real numbers, then $Z(J) = a(J) + J \cdot b(J)$ is called a double complex number which corresponds to the complex number pair $(Z_C, Z_H) = (a_C + \mathrm{i}b_C, a_H + \varepsilon b_H)$. Let $C(\lambda; J)$ and $S(\lambda; J)$ be the double real numbers generated by the sequences $\{a_n(\lambda)\} = \{\lambda^{2n}/(2n)!\}$ and $\{b_n(\lambda)\} = \{\lambda^{2n+1}/(2n+1)!\}$ respectively, and write $\exp(\lambda; J) = C(\lambda; J) + J \cdot S(\lambda; J)$, where λ is a real variable. $S(\lambda; J)$, $C(\lambda; J)$ and $\exp(\lambda; J)$ may be, respectively, called the double sine, the double cosine and the double complex exponential functions of λ. This is suitable because we have

$$\exp(\lambda; J) = \sum_{n=0}^{\infty} \frac{1}{n!} (\lambda J)^n \qquad C^2 - J^2 S^2 = 1 \tag{1}$$

and the function pairs

$$\begin{aligned}
[C_C(\lambda), C_H(\lambda)] &= (\cos \lambda, \cosh \lambda) \\
[S_C(\lambda), S_H(\lambda)] &= (\sin \lambda, \sinh \lambda).
\end{aligned} \tag{2}$$

Therefore, in the following we write simply $C(\lambda; J) = C[\lambda J]$, $S(\lambda; J) = S[\lambda J]$ and $\exp(\lambda; J) = \exp[\lambda J]$ or $e^{\lambda J}$.

For the case of $\eta = \mathrm{diag}(1, 1)$, the elements of the double group $\mathrm{SU}(\eta, J) = \mathrm{SU}(2, J)$ are all 2×2 double complex matrices M obeying

$$MM^{\dagger} = \mathbf{I} \qquad \det(M) = 1 \tag{3}$$

where $\mathbf{I}$ is the unit matrix, and the dagger denotes the double complex Hermitian conjugation. From $J = \mathrm{i}$ and $J = \varepsilon$, $\mathrm{SU}(2, J)$ denotes $\mathrm{SU}(2, C) = \mathrm{SU}(2)$ and $\mathrm{SU}(2, H)$ respectively, where C means ordinary complex, and H means hyperbolic complex. The bases of the Lie algebra of $\mathrm{SU}(2, J)$ can be taken as

$$\tau_1 = \begin{pmatrix} 0 & \frac{1}{2}J \\ \frac{1}{2}J & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} \frac{1}{2}J & 0 \\ 0 & -\frac{1}{2}J \end{pmatrix}. \tag{4}$$

Let operators $\mathscr{L}_{\pm} = \pm J_{\tau_2 - \tau_1}$ and $\mathscr{L}_3 = J^2_{\tau_3}$, then we have

$$[\mathscr{L}_3, \mathscr{L}_{\pm}] = J^3(\pm\mathscr{L}_{\pm}), [\mathscr{L}_+, \mathscr{L}_-] = 2J^3\mathscr{L}_3. \tag{5}$$

We consider formally a double real Hamiltonian

$$H(J) = \tfrac{1}{2}(p_1^2 - J^2 q_1^2) + \tfrac{1}{2}(p_2^2 - J^2 q_2^2) \tag{6}$$

and the symplectic form

$$\Omega(J) = \mathrm{d}q_1 \wedge \mathrm{d}p_1 + \mathrm{d}q_2 \wedge \mathrm{d}p_2 \tag{7}$$

on the phase space. Let

$$z_\alpha(J) = \frac{p_\alpha + Jq_\alpha}{\sqrt{2}} \qquad \bar{z}_\alpha(J) = \frac{p_\alpha - Jq_\alpha}{\sqrt{2}} \qquad (\alpha = 1, 2) \tag{8}$$

then

$$H(J) = \|z_1\|^2 + \|z_2\|^2 \qquad \Omega(J) = \frac{1}{J}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2) \tag{9}$$

where the modulus is $\|a + J\cdot b\|^2 = a^2 - J^2 b^2$. Let

$$\mathscr{G}_+(J) = z_1\bar{z}_2 \qquad \mathscr{G}_-(J) = \bar{z}_1 z_2 \qquad \mathscr{G}_3(J) = \tfrac{1}{2}(\|z_1\|^2 - \|z_2\|^2) \tag{10}$$

then the $\mathrm{SU}(2, J)$ algebra is realized in the sense of a Poisson bracket by $\mathscr{G}_3$ and $\mathscr{G}_{\pm}$, i.e.

$$\{\mathscr{G}_3, \mathscr{G}_{\pm}\} = J^3(\pm\mathscr{G}_{\pm}), \{\mathscr{G}_+, \mathscr{G}_-\} = 2J^3\mathscr{G}_3. \tag{11}$$

Now, the results in [3] can be directly extended to a double complex form as follows. Let

$$\begin{aligned} W_\alpha(J) &= \frac{1}{\sqrt{\gamma \sinh\gamma}} \frac{\sinh(\gamma\|z_\alpha\|^2)}{\|z_\alpha\|^2} z_\alpha \\ \bar{W}_\alpha(J) &= \frac{1}{\sqrt{\gamma \sinh\gamma}} \frac{\sinh(\gamma\|z_\alpha\|^2)}{\|z_\alpha\|^2} \bar{z}_\alpha \end{aligned} \tag{12}$$

where the parameter q must be taken as real and $\gamma = \ln q$. Next, let

$$\mathscr{G}'_+(J) = W_1\bar{W}_2 \qquad \mathscr{G}'_-(J) = \bar{W}_1 W_2 \qquad \mathscr{G}'_3(J) = \mathscr{G}_3. \tag{13}$$

By a direct calculation it can be verified that the double quantum group $SU_q(2, J)$ is realized in the sense of a Poisson bracket by $\mathscr{G}'_3$ and $\mathscr{G}'_\pm$, i.e.

$$
\begin{aligned}
&\{\mathscr{G}'_3 \mathscr{G}'_\pm\} = J^3(\pm \mathscr{G}'_\pm) \\
&\{\mathscr{G}'_+, \mathscr{G}'_-\} = J^3[2\mathscr{G}'_3] \equiv J^3 \frac{q^{2\mathscr{G}'_3} - q^{-2\mathscr{G}'_3}}{q - q^{-1}} = J^3 \frac{\sinh(\gamma 2\mathscr{G}'_3)}{\sinh \gamma}.
\end{aligned}
\tag{14}
$$

When $J = \mathrm{i}$, equation (14) changes into the result of [3].

As for the dynamics explanation, we consider the double real Hamiltonian

$$
H'(J) = \|W_1\|^2 + \|W_2\|^2 \tag{15}
$$

the double canonical equations of motion are

$$
\begin{aligned}
\frac{\mathrm{d}p_\alpha(J)}{\mathrm{d}t} &= \{p_\alpha, H'\} = J^2 \frac{1}{\gamma \sinh \gamma} \frac{\gamma \|z_\alpha\|^2 \sinh(2\gamma\|z_\alpha\|^2) - \sinh^2(\gamma\|z_\alpha\|^2)}{\|z_\alpha\|^4} q_\alpha(J) \\
\frac{\mathrm{d}q_\alpha(J)}{\mathrm{d}t} &= \{q_\alpha, H'\} = \frac{1}{\gamma \sinh \gamma} \frac{\gamma \|z_\alpha\|^2 \sinh(2\gamma\|z_\alpha\|^2) - \sinh^2(\gamma\|z_\alpha\|^2)}{\|z_\alpha\|^4} p_\alpha(J).
\end{aligned}
\tag{16}
$$

The exact solutions are

$$
p_\alpha(J) = A_\alpha C[J\omega_\alpha t] \qquad q_\alpha(J) = A_\alpha S[J\omega_\alpha t] \tag{17}
$$

where the double real functions $C[J_\lambda]$ and $S[J_\lambda]$ are defined by (1), A_α is the integral constant, and

$$
\omega_\alpha = \frac{2}{\gamma \sinh \gamma} \frac{\gamma A_\alpha^4 \sinh(\gamma A_\alpha^2) - 2\sinh^2(\frac{1}{2}\gamma A_\alpha^2)}{A_\alpha^4}. \tag{18}
$$

The solutions in (17) may yet be written as

$$
z_\alpha(J) = A_\alpha \exp(J\omega_\alpha t). \tag{19}
$$

When $J = \mathrm{i}$, equation (17) returns to the case in [3], i.e. it is a system of classical q-deformed harmonic oscillators

$$
p_{\alpha\mathrm{C}} = A_\alpha \cos \omega_\alpha t \qquad q_{\alpha\mathrm{C}} = A_\alpha \sin \omega_\alpha t. \tag{20}
$$

When $J = \varepsilon$,

$$
p_{\alpha\mathrm{H}} = A_\alpha \cosh \omega_\alpha t \qquad q_{\alpha\mathrm{H}} = A_\alpha \sinh \omega_\alpha t \tag{21}
$$

which is not a real system of oscillators. However, if the transformation $(p_\alpha, q_\alpha, t) \to (p'_\alpha, q'_\alpha, t') = (p_\alpha, \mathrm{i}q_\alpha, \mathrm{i}t)$ is allowed, then we obtain again the system (20).

As for $SU_q(1, 1, J)$, the case is similar, i.e.

$$
\begin{aligned}
&\hat{\eta} = \mathrm{diag}(1, -1) \\
&\hat{\tau}_1 = \begin{pmatrix} 0 & -\frac{1}{2}J \\ \frac{1}{2}J & 0 \end{pmatrix} \qquad \hat{\tau}_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad \hat{\tau}_3 = \tau_3 \\
&\hat{z}_\alpha(J) = \frac{p_\alpha - Jq_\alpha}{\sqrt{2}} \qquad \hat{\bar{z}}_\alpha(J) = \frac{p_\alpha + Jq_\alpha}{\sqrt{2}} \\
&\hat{\mathscr{G}}_+ = -\hat{z}_1 \hat{\bar{z}}_2 \qquad \hat{\mathscr{G}}_- = -\hat{\bar{z}}_1 \hat{z}_2 \qquad \hat{\mathscr{G}}_3 = -\tfrac{1}{2}(\|\hat{z}_1\|^2 + \|\hat{z}_2\|^2) \\
&\hat{W}_\alpha = W_\alpha(z_\alpha \to \hat{z}_\alpha) \\
&\hat{\mathscr{G}}'_+ = -\hat{W}_1 \hat{\bar{W}}_2 \qquad \hat{\mathscr{G}}'_- = -\hat{\bar{W}}_1 \hat{W}_2 \qquad \hat{\mathscr{G}}'_3 = \hat{\mathscr{G}}_3 \\
&[\hat{\mathscr{G}}'_3, \hat{\mathscr{G}}'_\pm] = J^3(\pm \hat{\mathscr{G}}'_\pm) \qquad [\hat{\mathscr{G}}'_+, \hat{\mathscr{G}}'_-] = J^3[-2\hat{\mathscr{G}}'_3] \\
&\ldots
\end{aligned}
\tag{22}
$$

and the double q-deformed oscillator solution is

$$z_\alpha(J) = A_\alpha \exp[(-1)^\alpha J\omega_\alpha t]. \tag{23}$$

By a similar method we can discuss the corresponding double Hopf algebra and the Yang–Baxter equation, etc.

The general meaning of the method in this letter is as follows. By the double complex method, it is possible that some solutions of quantum group problems by using the ordinary complex functions are doubled, then we can obtain the double solutions of the corresponding double quantum group problems. However we have proved [1] that a hyperbolic linear Lie group, generally, is locally isomorphic to a real linear Lie group or a direct product of two real linear Lie groups. In this case a problem about some kinds of quantum groups corresponding to real Lie groups may be changed into the corresponding problem concerning hyperbolic complex Lie groups; however this is just a half of the above double solution problem. For example, according to [1], SU(2, H) and SU(1, 1, H) are locally isomorphic to SL(2, R); therefore we have obtained a classical realization of $\mathrm{SL}_q(2, R)$ from the preceding results. In fact, if we force $J=1$ for all J's in (10)–(14), then we obtain just the realization. In addition, the above discussion also means that there is, in fact, some symmetry relation between the set of quantum groups corresponding to ordinary complex Lie groups and the set of some combination of quantum groups corresponding to real Lie groups.

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